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Reliability-Based Optimization: A Proposed Analytical-Experimental Study

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An analytical and experimental study for assessing the potential of reliability-based structural optimization is proposed and described. Competing designs obtained by deterministic and reliability-based optimization are compared. The experimental portion of the study is practical because the structure selected is a modular, actively and passively controlled truss that consists of many identical members, and because the competing designs are compared in terms of their dynamic performance and are not destroyed if failure occurs. The analytical portion of this study is illustrated on a 10-bar truss example. In the illustrative example, it is shown that reliability-based optimization can yield a design that is superior to an alternate design obtained by deterministic optimization. These analytical results provide motivation for the proposed analytical-experimental study, which is described elsewhere.

Nomenclature

c	= total cost of damping control system
c_0	= maximum allowable total cost of damping control system
$E(x_i)$	= mean values of gains of active members ($i = 1, 2$) and damping factors of passive members ($i = 3, 4$)
F_i	= failure mode associated with the ith vibratory mode
F_s	= system failure
$P(F_i)$	= probability of occurrence of failure mode F_i
$P(F_s)$	= probability of system failure
R_s	= system reliability, $1 - P(F_s)$
x_i	= normalized gains of active members 1 and 2 ($i = 1, 2$) and normalized damping factors of passive members 3 and 4 ($i = 3, 4$)
β_s	= system reliability (or safety) index
ζ_i	= damping factor of the ith vibratory mode
$\zeta_{0,i}$	= lowest acceptable damping factor for the ith vibratory mode
$\Phi(\cdot)$	= cumulative probability distribution function of a standard Gaussian random variable

Introduction

OVER the last two decades significant advancements have taken place in both the theory of and computational techniques used in structural reliability.[1-3] Analytical studies have demonstrated that reliability-based optimization can be more effective than classical deterministic optimization for designing aircraft structures because, for a minimum required reliability level, reliability-based optimization provides a more efficient design than deterministic optimization (e.g., Ref. 4). Tong and Yang[5] and Wang et al.[6] have also compared reliability-based and deterministic optimization. First they[5,6] minimized the weight of a 10-bar truss using deterministic constraints for static displacements and stress. Then they redesigned the truss using reliability-based optimization. Again they tried to minimize weight, but this time they considered that the properties of the truss were random and required that the reliability exceeded a minimum value. They found that the reliability-based design yielded a design that had higher reliability than the deterministic design, but the reliability-based design was also heavier. The main difference between Tong and Yang[5] and Wang et al.[6] on one hand and Yang et al.[4] on the other is that Tong and Yang[5] and Wang et al.[6] compared designs that had both different reliabilities and weights, whereas Yang et al.[4] compared the weights of two designs that had the same reliability. However, the superiority of probabilistic over classical deterministic methods has not been proven in real life design or by experiment. This is one of the critical reasons that reliability-based optimization has not been applied to aerospace structures. (In this paper, the terms reliability-based optimization and probabilistic optimization are used interchangeably to refer to optimization in which uncertainties and probability of failure are taken into account.)

Three critical issues involving the accuracy of and basic assumptions behind the methods for structural reliability assessment are as follows.

1) In reliability-based design, it is difficult to quantify all important uncertainties. For example, there is no consensus about how to quantify uncertainties that are a result of assumptions and simplifications in analysis procedures (modeling uncertainties). Moreover, it is difficult to collect sufficient data to develop accurate models of modeling uncertainties. Moore[7] provided examples of probabilistic analysis, which involved modeling uncertainties.

2) In many cases, there are significant errors in the assumed probability distributions of some random variables, such as, for example, the loads. Usually, the probability distribution of a load is estimated by analyzing sample values that are in the vicinity of the mean value and not in the right tail of the probability distribution. Since most failures occur when a load is large, however, it is the right tail of the probability distribution of a load that is important in reliability assessment. Similarly, it is difficult to determine the shape of the left tail of the probability distribution of material properties, which is also critical in reliability assessment. For some random variables, including the load and material properties just cited, a small error in the assumed distribution may cause a large error in the estimated probability of failure. The problem, which is often referred to as the tail sensitivity problem, is a serious consideration when assessing reliability.[2,8]

3) In many cases, the results of reliability analysis are very sensitive even to small changes in models of a structural configuration or of the applied loads. Pittaluga[9] has demonstrated how dramatically modeling errors affect the estimated probability of failure of a ship structure.

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As a result of the aforementioned problems, the probability of failure that is calculated in reliability analysis, and used in reliability-based design, can be significantly different from the actual failure probability. It is generally agreed that this nominal failure probability should be interpreted as a subjective measure of safety rather than as the actual failure probability. Consequently, it is important to answer the following question. If this nominal failure probability is used as a surrogate for the actual failure probability in reliability-based optimization will the resulting designs still be better than their deterministic counterparts?

If designers are to accept probabilistic methods as practical design tools, it is important to demonstrate experimentally that probabilistic methods can yield better designs than deterministic methods, despite the described difficulties. However, demonstrating the advantages of reliability-based optimization is not a simple task. To do so, many pairs of alternative structures obtained using reliability-based optimization and deterministic optimization must be constructed and tested, and their performance compared based on specified failure criteria. Depending on the failure criteria selected, it might be necessary to destroy a large number of sample structures to obtain a valid assessment. Since destroying a large number of structures would be impractical for this study, the structure and its failure modes were chosen in a way that failure would not imply its destruction.

The objectives of this paper are twofold. The first is to propose and describe an analytical-experimental study to assess the advantages of reliability-based optimization. The study will establish a practical procedure to compare reliability-based optimization with deterministic optimization. The second is to demonstrate with an analytical example that reliability-based optimization can lead to a design that is superior to a design obtained by deterministic optimization. This example will only illustrate the analytical part of the proposed study.

The key feature of the study is to use an actively and passively controlled modular truss structure and consider a large number of failure events that do not imply the destruction of the structure or its members. Such a failure event occurs, for example, when the damping ratio of a vibration mode falls below a specified value and/or when the vibration amplitude at some given location exceeds a maximum allowable value. During tests, the truss can be disassembled and reassembled after rearranging its members randomly. Thus, a large number of identical random samples of the same design can be tested at low cost. The large number of random samples, together with the large number of failure events, eliminates the factor of chance when comparing a probabilistic design with an alternative deterministic design and makes it possible to draw valid conclusions regarding the effectiveness of reliability-based optimization.

In the proposed study, two trusses are designed: one using deterministic optimization and the other using reliability-based optimization. In both cases the objective is to maximize safety; however, because the definition of safety differs in the two cases, the final designs are different. A large number of both designs are constructed and tested. The design that has the smaller number of failures is accepted as being better.

The general procedure for comparing deterministic optimization with reliability-based optimization is presented first. Then, the structure that is proposed for the study is described. Finally, an analytical example involving a 10-bar truss is used to provide motivation for the study and to illustrate some of its steps. When compared, the probabilistic design that is obtained in the example is superior to the deterministic design. The results of the experimental comparison of deterministic and probabilistic design are described in Maglaras et al.[10]

Description of Analytical-Experimental Study

The proposed study consists of an analytical part and an experimental part. In the analytical part, two trusses are designed: one using reliability-based optimization and the other using deterministic optimization. In each case, the objective is to maximize safety without exceeding its specified cost. In deterministic optimization, safety is quantified by the margin of safety; in reliability-based optimization, safety is quantified by the system reliability. After the two trusses are designed, their system reliabilities are evaluated and compared analytically.

In the experimental part, many samples of the probabilistic and deterministic designs are tested to determine their dynamic performance. Then, the number of failure events for the two types of designs are compared. Failure events are defined by unacceptable dynamic behavior. Examples include cases where the damping ratio of any of the modes is smaller than a prescribed minimum value, or the vibratory amplitude of any node is larger than a prescribed maximum value. In the tests, if the percentage of failures for the probabilistic designs is smaller than the percentage of failures for the deterministic designs, it is concluded that the probabilistic designs are safer. Moreover, because both designs have the same cost, the reliability-based design would be a more effective approach for structural design.

Analytical Part of Study, Probabilistic and Deterministic Design

Frangopol[11] reviewed several formulations of system reliability optimization problems. A system reliability-based optimization problem, which considers only one objective, seeks to minimize one of the following quantities: 1) the total expected cost, which is the sum of the initial cost of a structure and the expected cost of failure, which is proportional to the probability of failure; 2) the initial cost of a structure so that system reliability exceeds a minimum value; and 3) the system probability of failure so that cost is kept below a maximum value.

The first formulation has been rarely used. In this study we selected the third formulation, which maximizes safety for given cost, because it allows for an easier experimental comparison of probabilistic and deterministic optimization than the second formulation.

When a structure is designed probabilistically to maximize its safety, the objective is to minimize the probability of system failure $P(F_s)$. This is done subject to the requirement that the utilized resources (cost, weight, energy expended) do not exceed the allocated resources. It is assumed that the system fails if its dynamic behavior is unacceptable.

For example, failure of the system can be defined as the event in which the damping ratio of any vibratory mode falls below a lowest acceptable value for that mode. Specifically, the failure mode associated with the ith vibratory mode F_i is an event that occurs if the damping factor of the ith vibratory mode ζ_i becomes less than the lowest acceptable damping for that mode, $\zeta_{0,i}$.

That is,

$$F_i: \zeta_i < \zeta_{0,i} \qquad i = 1, \ldots, n \tag{1}$$

Because of this definition, there is a one-to-one correspondence between failure modes and vibratory modes. Therefore, the same subscript is used to specify the failure mode and vibratory mode number. In general, there is no one-to-one correspondence between failure modes and vibratory modes.

When a structure is designed deterministically to maximize its safety, the objective is to maximize the margin against failure. Using the preceding example and notation, the margin of safety for failure mode i is defined to be $\zeta_i - \zeta_{0,i}$. Thus, the objective is as follows.

Maximize:

$$\min(\zeta_i - \zeta_{0,i}) \qquad i = 1, \ldots, n \tag{2}$$

such that the utilized resources do not exceed the allocated resources. It is assumed that the same resources are allocated in deterministic and probabilistic design.

After the two alternate trusses are designed, their probabilities of failure are evaluated analytically and compared. If the probabilistic design has a significantly lower failure probability than the deterministic design (for example, if it is 50% of the failure probability of the deterministic design), then it is anticipated that the probabilistic design will perform better than the deterministic design in an experiment in which many samples of the two designs are tested. Since it is difficult to measure small differences in probabilities of failure, the experimental part of the study is undertaken if and only if the failure probability of the probabilistic design is significantly lower than that of the deterministic design. Otherwise, the design requirements are redefined (by changing the required damping factor, allowable vibration amplitude, etc.) to produce two new designs

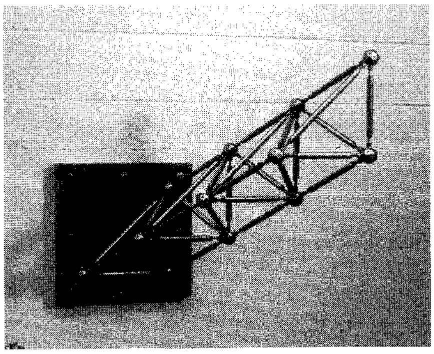


Fig. 1 Typical experimental truss, without dampers.

that do have significantly different failure probabilities. These new designs can be used in the experimental part of the study.

Experimental Part of Study

Here we describe only the general properties of a structure that can be used for experimental comparison of reliability-based design and deterministic design. This comparison has been completed and described in Maglaras et al.[10]

Testing involves a large number of pairs of structures, each pair consisting of a design obtained by deterministic optimization and a design obtained by reliability-based optimization. The number of structures of each type that fail is recorded. If the mathematical models that describe the uncertainties and structural response are sufficiently accurate, then it is likely that more deterministic than probabilistic designs will fail. This will demonstrate that the probabilistic design is more reliable than its deterministic counterpart.

Description of Experimental Structure

A modular truss, as shown in Fig. 1, was selected for the study because, as explained in the Introduction, it is suitable for the experimental procedure. The truss consists of two sets of identical struts bolted together using joints. Concentrated masses can be attached to the joints. The dynamic behavior of the truss is controlled by using active struts and/or passive dampers. The truss in Fig. 1 has no dampers.

Active struts consist of piezoelectric sensors and actuators with integral control. This type of active member is described by Preumont et al.[12] and Ponslet et al.[13]

Passive dampers can be constructed by coating struts with viscoelastic material, which allows the members to absorb energy. The behavior of the passive members is nonlinear because both the damping factor and the stiffness depend on the displacement amplitude. Moreover, the damping factor depends on the temperature and the frequency. Jones[14] reviewed the characteristics of viscoelastic materials when used for damping applications. Other passive damping concepts will also be explored.

There is uncertainty in the damping ratio and stiffness of both the passive and active members, because of variability between samples and because of the dependence of these quantities on both the deformation and temperature of these members. In addition, the dimensions of the structural members, the material properties, and the weights of the concentrated masses vary randomly because of manufacturing imperfections.

Example: 10-Bar Truss

The following analytical example illustrates the procedure described earlier and demonstrates that reliability-based optimization can produce a superior design. In the example, two 10-bar trusses are designed and compared. One truss is designed using deterministic optimization and the other using reliability-based optimization. In both cases the objective is to maximize safety. Topics presented include the optimization procedures, the resulting two designs, and the performance of the two designs.

Each of the two alternative designs was assumed to fail when the damping ratio of any of the first four natural vibration modes fell below a specified value. Two active and two passive members were used to control damping. The truss configuration and the locations of active and passive members and concentrated masses are shown in Fig. 2. Each active member consisted of a piezoelectric stack to provide actuation and a force transducer to provide sensing. An integral control law was used. The truss properties that were used in the optimization are presented in Table 1. The members were modeled as bars that resist extension and compression, not bending. In some modes, especially the higher ones, bending can be important. In these modes the flexibility of the joints can also be important.

Table 1 Properties of 10-bar truss and design requirements

Element cross-sectional area, m^2	$0.132 \cdot 10^{-3}$
Length of short members, m^2	9
Young's modulus, N/m^2	$70 \cdot 10^9$
Density, kg/m^3	$3 \cdot 10^3$
Concentrated masses, kg	10
Nominal gains of active members, $m/N \cdot s$	$0.5 \cdot 10^{-4}$
Nominal damping factors of passive members, $N \cdot s/m$	200
Lowest acceptable damping ratio, %	2.5
Maximum allowable cost, \$[a]	249
Cost of active members, \$	100/unit of normalized gain
Cost of passive members, \$	20/unit of normalized damping factor

[a]Values selected for the unit costs are arbitrary and are used for illustrative purposes only.

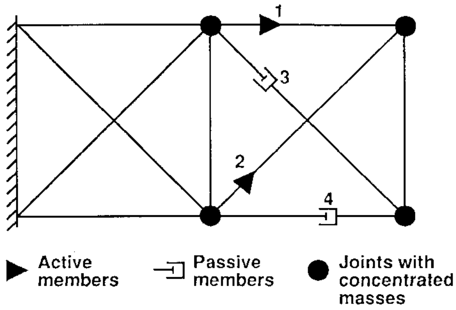


Fig. 2 Example using 10-bar truss; numbers indicate locations of the corresponding design variables.

Optimization Procedures and Results

In the following section, the deterministic and probabilistic optimization procedures are summarized and the final designs are presented. The two optimization procedures differ only in the objective function. In the deterministic case, the objective is to maximize the minimum margin of safety. In the probabilistic case, the objective is to minimize the probability of system failure. In both cases, the algorithm that was used for the optimization was an extended interior penalty function technique incorporated in the code NEWSUMT-A.[15] This solves linear and nonlinear optimization problems with both equality and inequality constraints. The user provides subroutines for objective function and constraint evaluation. For example, in this problem, we provided subroutines that calculated the damping ratios of the modes (deterministic and probabilistic optimization) or the system probability of failure (probabilistic optimization). NEWSUMT-A finds the values of the design variables that minimize the objective function and satisfy the constraints. The basic algorithm used in NEWSUMT-A is the sequence of unconstrained minimizations using Newton's method with approximate derivatives for unconstrained function minimizations.

Deterministic Optimization

The optimizer tries to maximize the lowest margin of safety. The gains of the active members and the damping factors of passive members were assumed to be deterministic. Failure was defined as a damping ratio that was less than a specified allowable value for the first four vibration modes. Because the allowable value was the same (2.5%) for each vibration mode, the optimization procedure reduces to maximizing the lowest damping ratio. During the optimization, the total cost associated with the active and passive members was not allowed to exceed the maximum allowable cost. The optimization is defined formally as

Table 2 Deterministic optimum for 10-bar truss example

Member type/member number	Normalized gain or damping factor	Cost, $
Active/1	1.848	184.8
Active/2	0.530	53.0
Passive/3	0.559	11.18
Passive/4	0.0	0.0

Maximize:

$$\min(\zeta_1, \ldots, \zeta_4) \tag{3}$$

Subject to:

$$c = 100(x_1 + x_2) + 20(x_3 + x_4) \leq c_0$$

Design variables were the normalized gains x_1 and x_2 of the active members and the normalized damping factors x_3 and x_4 of the passive dampers. The subscript on x indicates the member number. Locations of the active and passive members are shown in Fig. 2. Because the damping ratios are nonlinear functions of the gains and damping factors Eq. (3) is a nonlinear optimization problem.

Table 2 presents the values of the gains and the damping factors together with the costs of each active and passive member for the deterministic optimum. At the optimum, the damping ratio of the third mode is 3.1% and the damping ratios of the other three modes are all 3.0%.

To test whether the deterministic optimum in Table 2 was the global optimum we solved the deterministic optimization problem starting from several initial designs, including the optimum reliability-based design presented in the next section. In all cases, the optimization algorithm converged to the deterministic optimum shown in Table 2.

Reliability-Based Optimization

Reliability-based optimization accounts for uncertainties. There are many types of uncertainties, such as uncertainty in the flexibility of the joints. These uncertainties can be classified into two categories: modeling and random. Modeling uncertainties are the result of simplifications in modeling and analyzing the structure. Random uncertainties are the result of variability between samples and variability in the conditions under which the members operate (temperature, amplitude of vibration) during the same experiment or from one experiment to another.

Only random uncertainties in the gains and the damping factors were considered for this example. The gains and the damping factors were assumed to be independent, Gaussian random variables whose coefficients of variation were equal to 10%. Subsequently we examine the effect of errors in the probability distributions of the random variables on the system failure probability of the optimum.

After the uncertainties were defined, the truss was optimized by minimizing the probability of system failure $P(F_s)$. As in the deterministic design, the total cost associated with the active and passive members was not allowed to exceed the maximum allowable cost. Formally, the following optimization problem was solved.

Minimize:

$$P(F_s) \tag{4}$$

Subject to:

$$c = 100[E(x_1) + E(x_2)] + 20[E(x_3) + E(x_4)] \leq c_0$$

Calculating $P(F_s)$ can be a formidable task that involves calculation of the probabilities of failure corresponding to each vibratory mode and the joint probabilities of failure corresponding to all possible combinations of vibratory modes.[1,2] Fortunately, there are approximate procedures for making that calculation that are, in most cases, reasonably accurate. Examples include first- and second-order, second moment methods for calculation of the probabilities of failure corresponding to each vibratory mode and combinations of these modes, and the first- and second-order Ditlevsen bounds for derivation of $P(F_s)$ from the given probabilities.[1,2,16] In the present study, a second moment method and the second-order, upper Ditlevsen bound were used to estimate the system failure probability.

The maximum allowable cost c_0 was the same as that used in the deterministic optimization. Design variables were the mean values of the normalized gains $E(x_1)$ and $E(x_2)$ of the active members (members 1 and 2) and the mean value of the normalized damping factor $E(x_3)$ of one of the passive dampers (member 3). The damping factor x_4 of the other passive damper (member 4) was assumed to be zero for the following two reasons. First, in the deterministic design x_4 was found to be zero. Second, as is shown in the following section, the system reliability index for the deterministic design was found to be insensitive to x_4.

Table 3 presents the mean values of the gains and the damping factors together with the costs of each active and passive member for the reliability-based optimum.

Table 3 Reliability based optimum for 10-bar truss example

Member type/member number	Mean values of normalized gain or damping factor	Cost, $
Active/1	1.941	194.1
Active/2	0.413	41.3
Passive/3	0.675	13.5
Passive/4	0.0	0.0

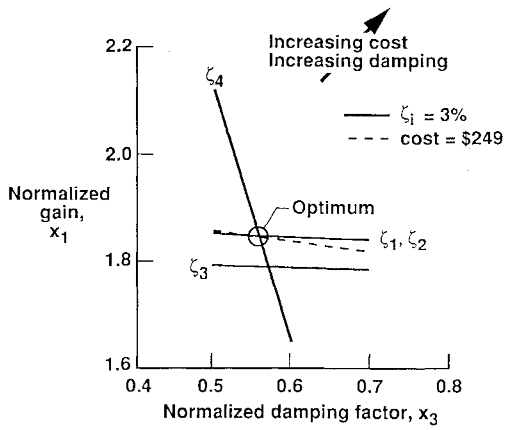


Fig. 3 Design space for deterministic optimum; each curve, except cost, defines a 3% damping ratio for the corresponding vibration mode.

Description and Comparison of Deterministic and Probabilistic Designs

Deterministic Design

The design space for the deterministic optimum is shown in Fig. 3. Recall that the objective was to maximize the minimum ζ_i subject to an upper limit on cost of $249 and that the optimum was found using mathematical programming techniques. In Fig. 3, cost and damping ratios increase toward the upper right. The optimum design is bounded from above by the line defined by cost = $249. Moving to the left along that line causes ζ_4 to decrease, whereas moving to the right causes ζ_1 and ζ_2 to decrease. At the optimum, the damping ratio of the third vibration mode is 3.1%; for the first, second, and fourth modes, the damping ratios are all 3.0%. Note that, according to the deterministic analysis, mode 3 is less important than the other modes because its damping ratio is highest.

The effect of the normalized gain of active member 1 on the damping ratios of the four vibration modes is depicted in Fig. 4. Modes 2 and 3 are the most sensitive, whereas mode 4 is the least sensitive. As in Fig. 3, the deterministic optimum is at the intersection of the lines representing modes 1, 2, and 4. The damping ratio of modes 1, 2, and 4 is 3% at the intersection. The probability of system failure for the deterministic optimum was estimated. For each mode, failure was defined to occur when the damping ratio of that mode was less than 2.5%. System failure was defined to occur when any of the damping ratios of the first four vibration modes fell below 2.5%. The remainder of the section presents the results of probabilistic analysis performed on the deterministic design.

Table 4 Probabilities of failure of modes, system probability of failure, and system reliability index of deterministic design

$P(F_1)$, %	$P(F_2)$, %	$P(F_3)$, %	$P(F_4)$, %	$P(F_s)$, %	β_s
1.06	4.65	2.28	0.23	4.81	1.664

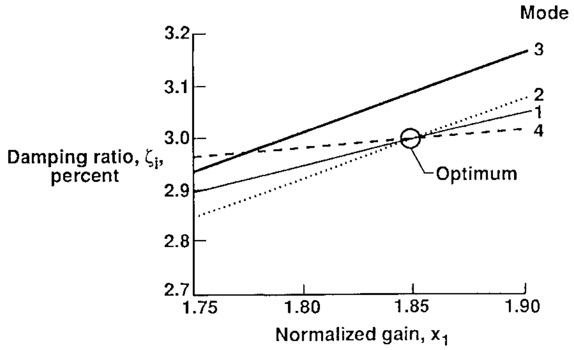


Fig. 4 Deterministic optimum; effect of normalized gain x_1 on damping ratios of first vibration modes.

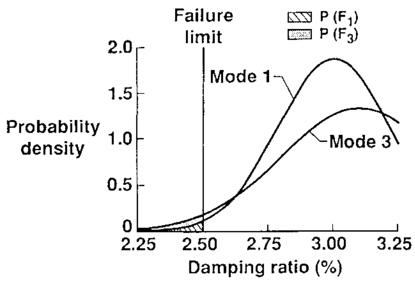


Fig. 5 Comparison of probabilities of failure of modes 1 and 3.

Table 4 presents the results of a reliability analysis of the deterministic design. Model 2 is the most critical mode, and mode 3 is the second most important mode. The probability of failure of mode 4 is small compared to those of the other modes. The safety of the design is indicated by the system failure probability, $P(F_s) = 4.81\%$. Another measure of the safety of the design, the system reliability index, is also given in Table 4. The system reliability index β_s is defined by

$$R_s = \Phi(\beta_s) \tag{5}$$

The system failure probability was also estimated using Monte Carlo simulation with 10,000 samples. This probability was found to be 5.23%, which is approximately 9% larger than the probability estimated using the combination of second moment method and Ditlevsen bound [$P(F_s)$ in Table 4].

Regarding the relative importance of the modes to system failure, deterministic analysis and probabilistic analysis reach different conclusions. Specifically, deterministic analysis indicates that mode 3 is the least important of the four modes (Fig. 3). However, a probabilistic analysis of this deterministic design indicates that mode 3 is the second most important mode. Specifically, according to Table 4, the probability that the damping ratio is less than 2.5% (the value that corresponds to failure) is more than twice as great in the third vibration mode as it is in the first vibration mode. Therefore, mode 3 is actually more important than mode 1. Figure 5 helps explain the discrepancy between the conclusions of probabilistic and deterministic analysis. According to this figure, although the average damping ratio of mode 3 is larger than that of mode 1, the standard deviation of the damping ratio of mode 3 is so large compared to that of mode 1 that mode 3 has higher failure probability than mode 1. The areas that correspond to the failure probability of modes 3 and 1 are indicated by the dotted and cross-hatched regions, respectively. The dotted region includes the cross-hatched region.

Table 5 Deterministic design: logarithmic sensitivity derivatives of reliability indices for failure modes with respect to standard deviations of gains and dampers

Member number/mode	1 (active)	2 (active)	3 (passive)	4 (passive)
1	0.89	0.11	0	0
2	1.0	0	0	0
3	1.0	0	0	0
4	0.18	0.44	0.37	0

Table 6 Probabilities of failure of modes, system probability of failure, and system reliability index of reliability-based design

$P(F_1)$, %	$P(F_2)$, %	$P(F_3)$, %	$P(F_4)$, %	$P(F_s)$, %	β_s
2.2	2.0	1.0	0.23	2.739	1.921

The logarithmic sensitivity derivatives of the reliability indices for the four failure modes with respect to the standard deviation of the gains and damping factors are presented in Table 5. As used herein, the logarithmic sensitivity derivative of $f(x)$ with regard to x is given by $x(\mathrm{d}/\mathrm{d}x)(\log_e f)$, which is $x(f'/f)$. This derivative gives the relative change in f caused by a unit relative change in x, or it can be interpreted as the percent change in f caused by a 1% change in x. If f is the reliability index and x_i are the standard deviations of the random variables, then the sum of the logarithmic derivatives is unity, as shown in Table 5. The following observations are made.

1) The reliability index for failure mode 1 is sensitive only to the uncertainties in the gains of the active members (members 1 and 2).

2) The reliability indices of failure modes 2 and 3 are sensitive only to the gain of member 1.

3) Both active members and the passive damper (member 3) provide damping to vibration mode 4. However, the reliability index for failure in this mode is more sensitive to the uncertainties in the gain of the second active member and the passive damper than to the uncertainty in the gain of the first member.

4) The second passive damper (member 4) is unimportant; none of the failure modes is sensitive to the damping factor of this member.

Probabilistic Design

Table 6 presents the probabilities of failure of the four modes and the system failure probability of the probabilistic design. The system reliability index of the probabilistic design was calculated to be 1.921, which corresponds to a probability of failure of about 2.7%. The failure probability was also found to be about 2.7% using Monte Carlo simulation. By comparing Table 6 with Table 4, it can be seen that the ranking of the failure modes is different. For example, the probabilistic design is more likely to fail under mode 1, whereas the deterministic design is more likely to fail under mode 2. It is also observed that the probabilities of failure of the modes of the deterministic design differ more than those of the probabilistic design.

Figure 6 shows how the probability of system failure and the cost vary with the mean values of both the gain x_1 of active member 1 and the damping factor x_3 of passive member 3. In Fig. 6, the mean value of the gain, x_2, of active member 2 is equal to 0.413, which is the value that x_2 takes at the optimum. The solid curves correspond to constant values of the probability of system failure $P(F_s)$. The dashed lines correspond to constant values of total cost c. The reliability-based optimum is indicated by the triangular symbol. At the optimum, the line that corresponds to a cost of $249 is tangent to the curve that corresponds to a probability of failure of about 2.7%. Starting at the optimum, a move to the left or right along that $249 line causes the probability of system failure to increase. This confirms the fact that, out of all the designs that cost $249, the probabilistic optimum (triangular symbol) has the smallest probability of system failure.

Figure 7 shows how the safety of optimum probabilistic designs varies with the maximum allowable cost. The vertical scale at the left is the system reliability index. The vertical scale at the right is

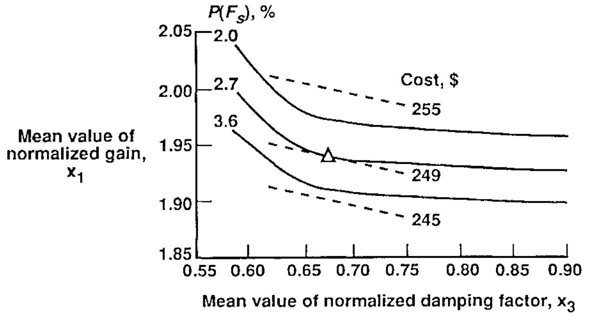


Fig. 6 Design space for reliability-based optimum; curves define constant values of system failure probability and total cost; $x_2 = 0.413$, $x_4 = 0.0$: ——, constant failure probability; ---, constant cost; and △, optimum for cost = \$249.

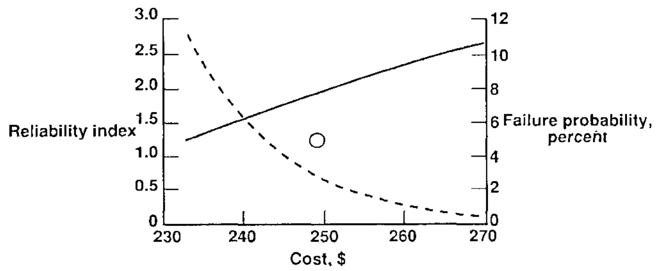


Fig. 7 System reliability index β_s and system probability of failure $P(F_s)$ vs total cost for optimum probabilistic designs; circular symbol indicates failure probability of deterministic optimum: ——, reliability index; ---, failure probability; and ○, deterministic optimum.

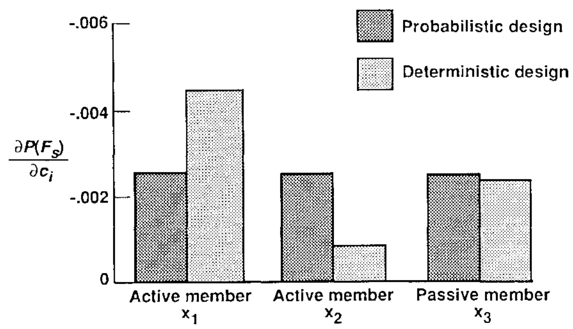


Fig. 8 Derivatives of system failures probability with respect to member cost and deterministic optima.

the probability of system failure. The horizontal scale is the maximum allowable cost. Over the range of costs shown, the safety of the system increases as the allowable cost increases. The circle indicates the failure probability (rather than the reliability index) of the deterministic optimum.

Comparison

In the reliability-based optimization procedure used herein, the probability of system failure is minimized subject to the requirement that the cost is less than an upper limit. The cost is the sum of the costs of the active members and the passive dampers. Therefore, the optimality criterion for the probabilistic optimum is

$$\frac{\partial P(F_s)}{\partial c_i} = \text{const} \qquad \text{for} \qquad i = 1, \ldots, 3 \tag{6}$$

where c_i is the cost of the ith member. According to this criterion, at the optimum, the three sensitivity derivatives in Eq. (6) are all equal. The three sensitivity derivatives for the deterministic and probabilistic optima are presented in Fig. 8.

For the deterministic optimum, the sensitivity derivatives are not equal; therefore, the deterministic optimum violates the optimality criterion. The sensitivity derivative with respect to the cost of member 1 (the first active member) is largest, and the sensitivity derivative with respect to the cost of member 2 is the smallest. Moreover, the sensitivity derivatives of member 3 of both probabilistic and deterministic designs are almost the same. Therefore, by increasing the gain of member 1 and reducing the gain of the other active member and/or the damping of the passive member, the safety of the system can be improved without exceeding the budget.

In contrast, for the reliability-based optimum, the sensitivity derivatives are almost identical. This means that, as long as the total cost is fixed, the safety cannot be increased by changing the gains and the damping factor.

Table 7 Comparison of the deterministic and probabilistic optimum

	Deterministic optimum	Probabilistic optimum
x_1	1.8483	1.941
x_2	0.53	0.413
x_3	0.56	0.675
Cost, \$	249.0	249.0
$P(F_s)^a(\beta_s)^a$	0.048(1.664)	0.027(1.921)
$P(F_s)^b(\beta_s)^b$	0.052(1.628)	0.027(1.927)
$\min(\zeta_1, \ldots, \zeta_4)$	$\zeta_1 = \zeta_2 = \zeta_4 = 0.030$	$\zeta_1 = 0.02954$

[a]Estimated using the second-order, upper Ditlevsen bound.
[b]Estimated using Monte Carlo simulation with 10,000 replications.

Table 7 compares the probabilistic and deterministic optima. Specifically, for each optimum design, it presents the mean values of the gains of the active members and the damping factor of the passive member, the cost, the system probability of failure, and the lowest damping ratio of the four vibratory modes of each design, calculated assuming that the gains of the active members and the damping factor of the passive member are deterministic and are equal to their mean values. Note that deterministic optimization seeks to maximize the value of the lowest damping ratio. Although the two optima cost the same, each design is better according to its own criterion. The probabilistic optimum is considerably safer than the deterministic optimum because the probability of failure of the probabilistic optimum is about 56% of that of the deterministic optimum. On the other hand, if one ignores uncertainties and calculates the lowest damping ratio of each design one will find that deterministic design is better. Of course, the ultimate test is how a design behaves in the field. If the model of the truss and the model of the uncertainties are accurate, then, in an experiment, it is likely that the probabilistic design will perform better than the deterministic design because it has lower probability of failure.

The reliability indices of the two optima were also evaluated using Monte Carlo simulation. The results, which are also presented in Table 7, agree with those obtained using the combination of second-moment methods and the second-order, upper Ditlevsen bound.

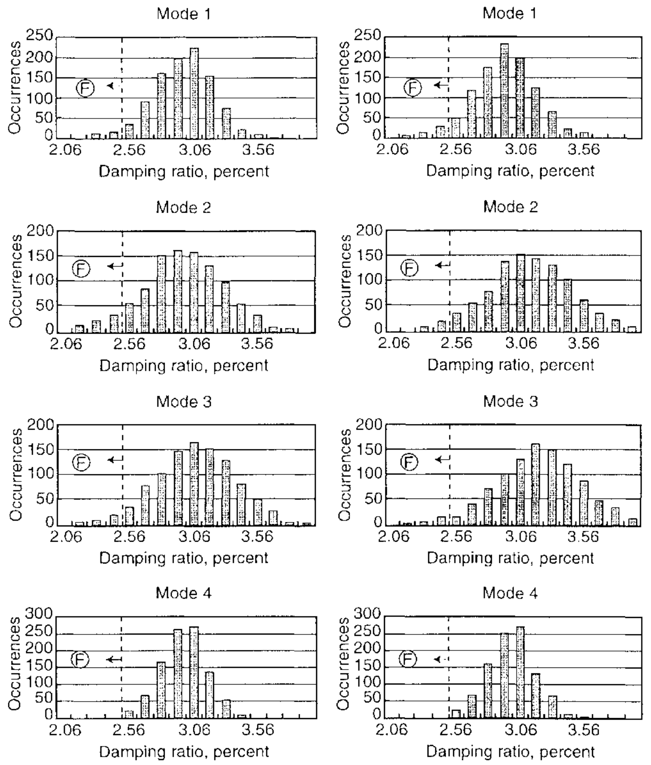
Figure 9 compares the histograms of the damping ratios of the four modes of the deterministic and probabilistic designs. These histograms were calculated using Monte Carlo simulation (1000 sample points, whereas 10,000 sample points were used earlier). The probability of failure of each mode is the number of sample points that lie to the left of the minimum damping ratio, which is 2.5% (indicated by the dashed lines in Fig. 9). The deterministic design is more likely to fail under the second mode than any other mode. The probabilistic design trades a small amount of reliability of the first mode (failure probability increases from 1.1 to 2.2%; Tables 4 and 6) for a significant increase in the reliability of mode 2 (failure probability reduces from 4.65 to 2.0%; Tables 4 and 6) by reducing the mean value of the damping ratio of the first mode and increasing the mean value of the damping ratio of mode 2 (Fig. 9). This results in a design that has significantly lower system probability of failure than the deterministic design and costs the same.

Although the probability of failure of the probabilistic design is roughly 50% of that of the deterministic design, the difference of these probabilities is too small to be measured experimentally at low cost. Specifically, we expect that we will need about 500 experiments to validate the superiority of probabilistic optimization. Therefore, we need to redefine the optimization problems so that the difference in the failure probabilities of the alternate optimum designs is large enough to be measured using few experiments (e.g., 30). However, the difference in the probabilities of failure in Table 7 was deemed

Table 8 Effect of errors in the assumed probability distribution of x_1 on the probabilities of failure of the probabilistic and deterministic design

Probability distribution of x_1	Probability of failure of deterministic design	Probability of failure of probabilistic design
Uniform	0.016	0.005
Lognormal	0.043	0.025
Normal	0.052[a]	0.027[a]

[a]Estimated using 10,000 replications.



Deterministic design **Reliability-based design**

Fig. 9 Deterministic and probabilistic designs: histograms of damping ratios of modes.

sufficiently large for analytically demonstrating the advantages of probabilistic design. Maglaras et al.[10] describe a methodology for determining the values of the problem parameters (e.g., standard deviation of damping factor of passive member, minimum damping ratios) so that the difference between the probabilities of failure of the probabilistic design and deterministic design is large.

Effect of Errors in the Assumed Probability Distributions of the Random Variables

In real life there is rarely enough information for finding the type of the probability distributions of the random variables. Fox and Safie[17] have shown that the type of the assumed probability distribution of the random variables significantly affects the estimated probability of failure and that the normalcy assumption is unconservative. They have also issued guidelines for selecting the probability distribution of a random variable, which leads to estimates of the failure probability that are conservatively consistent with the information available. To examine the effect of such errors on the effectiveness of the probabilistic design, we assumed that in reality the gain of the first member x_1 was either uniform or lognormal. (We present results in cases where only the assumed probability distribution of x_1 was different than the actual one because we found that the types of probability distributions of the other random variables did not affect the probability of system failure.) The coefficient of variation of x_1 was assumed to be 10%. We calculated the probabilities of failure of the deterministic and probabilistic designs using the actual probability distribution and using Monte Carlo simulation with 1000 replications. The results are shown in Table 8. It is observed that in both cases, where x_1 is uniform or lognormal in reality, the probabilistic design is still safer than the deterministic design. However, we cannot draw any general conclusions about the sensitivity of the optimum to the normalcy assumption because Table 8 presents only two cases.

Conclusions

This report has proposed and described an analytical and experimental study for comparing reliability-based optimization with deterministic optimization. The study involves designing, testing, and comparing alternative structural designs obtained by probabilistic and deterministic methods. The experimental portion of the study is practical because of the following two important features:

1) The study uses a modular truss that consists of many identical members. After the truss is tested, it can be disassembled and its members rearranged. Then it can be reassembled and retested, assuming that no damage is inflicted to the truss during testing.

2) The designs are compared by testing their dynamic performance. Failure is defined to occur if the dynamic performance is unacceptable. Thus, failure does not imply destruction of the structure.

These two features allow many random samples of the same structure to be constructed and tested using a small number of members.

The following are the conclusions from an analytical example presented herein:

1) In the example, reliability-based optimization produced a design that was more reliable than, and cost the same as, the corresponding deterministic optimum because reliability-based optimization accounted for some of the uncertainties in a rational way and reliability-based optimization accounted for the sensitivity of the cost and performance of the system with respect to these uncertainties.

2) Deterministic analysis incorrectly ranked the failure modes in terms of their importance to system reliability.

Acknowledgments

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References

[1]Madsen, H. O., Krenk, S., and Lind, N. C., *Methods of Structural Safety*, Prentice–Hall, Englewood Cliffs, NJ, 1986.

[2]Melchers, R. E., *Structural Reliability Analysis and Prediction*, Ellis Horwood, London, 1987.

[3]Cruse, T. A. (ed.), *Reliability Technology 1992*, AD-Vol.28, American Society of Mechanical Engineers, New York, 1992.

[4]Yang, J. S., Nikolaidis, E., and Haftka, R. T., "Design of Aircraft Wings Subjected to Gust Loads: A System Reliability Approach," *Computers and Structures*, Vol. 36, No. 6, 1990, pp. 1057–1066.

[5]Tong, T. Y., and Yang, R. J., "An Advanced Reliability Based Optimization Method for Robust Structural System Design," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 34th Structures, Structural Dynamics, and Materials Conference* (La Jolla, CA), AIAA, Washington, DC, 1993, pp. 1198–1206.

[6]Wang, L., Grandhi, R. V., and Hopkins, D. A., " Structural Reliability Optimization Using an Efficient Safety Index Calculation Procedure," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 35th Structures, Structural Dynamics and Materials Conference* (Hilton Head, SC), AIAA, Washington, DC, 1994, pp. 853–863.

[7]Moore, N. R., Ebbeler, D. H., Newlin, L. E., Sutharshana, S., and Creager, M., "An Improved Approach for Flight Readiness Certification—Methodology for Failure Risk Assessment and Fatigue Failure Mode Case Study Examples," Jet Propulsion Lab., JPL 92-15, California Inst. of Technology, Pasadena, CA, June 1992.

[8]Ben-Haim, Y., and Elishakoff, I., *Convex Models of Uncertainty in Applied Mechanics*, Elsevier, Amsterdam, 1990.

[9]Pittaluga, A., "Reliability Based Ship Design in the 90s: Realistic Scenario or a Dream," *Registro Italiano Navale, Technical Bulletin*, No. 100, Genova, Italy, Oct. 1987.

[10]Maglaras, G., Ponslet, E., Haftka, R. T., Nikolaidis, E., Sensharma, P., and Cudney, H. H., "Analytical and Experimental Comparison of Probabilistic and Deterministic Optimization," *AIAA Journal* (to be published).

[11]Frangopol, D. M., "Reliability-Based Optimum Structural Design," *Probabilistic Structural Mechanics Handbook*, edited by C. R. Sundararajan, Chapman and Hall, New York, 1995, pp. 352–387.

[12]Preumont, A., Dufour, J. P., and Malekian, C., "Active Damping by a Local Force Feedback with Piezoelectric Actuators," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 32nd Structures, Structural Dynamics, and Materials Conference* (Baltimore, MD), AIAA, Washington, DC, 1991, pp. 1879–1887.

[13]Ponslet, E., Haftka, R. T., Hallauer, W. L., and Cudney, H. H., "Desensitizing Structural-Control Design," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 1, 1994, pp. 175–180.

[14]Jones, D. I. G., "Viscoelastic Materials for Damping Applications," *Damping Applications for Vibration Control*, AMD-Vol. 38, American Society of Mechanical Engineers, New York, 1980.

[15]Grandhi, R. V., Thareja, R., and Haftka, R. T., "NEWSUMT-A: A General Purpose Program for Constrained Optimization Using Constraint Approximations," *Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 107, No. 1, 1985, pp. 94–99.

[16]Ditlevsen, O., "Narrow Reliability Bounds for Structural Systems," *Journal of Structural Mechanics*, Vol. 7, 1979, pp. 453–472.

[17]Fox, E. P., and Safie, F., 1992, "Statistical Characterization of Lifetime Driver Distributions for a Probabilistic Life Analysis," AIAA Paper 92-3414, July 1992.